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Topological spaces close to being σ-compact

Abstract. All topological spaces are assumed to be metrizable and separable. The subsequent property of a topological space X was introduced in [1] by W. Hurewicz at the beginning of the 20-th century as a cover counterpart of the σ-compactness: for every sequence $(u_n)_{n\in\omega}$ of open covers of X there exists a sequence $(v_n)_{n\in\omega}$ such that each v_n is a finite subset of u_n and $\{\cup v_n : n \in \omega\}$ is a γ-*cover* of X, which means that the family $\{n \in \omega : x \notin \cup v_n\}$ is finite for every $x \in X$. Hurewicz conjectured that this property coincides with the σ-compactness. But as it was shown in [2], there exists a ZFC-example of an uncountable space X with the Hurewicz property and without uncountable compact subspaces. It was also asked there whether every such a space X has the subsequent formally stronger property $S_1(\Gamma, \Gamma)$: for every sequence $(u_n)_{n\in\omega}$ of open γ-covers of X there exists a γ-cover $\{U_n : n \in \omega\}$ of X such that $U_n \in u_n$.

The following theorem implies that the negative answer onto this question is consistent.

Theorem 1. *Under CH there exists an uncountable topological space X with the subsequent properties:*

(1) *X contains no uncountable compact subspaces;*
(2) *X^n has the Hurewicz property for every $n \in \mathbb{N}$;*
(3) *X does not have the property wQN (which follows from $S_1(\Gamma, \Gamma)$).*

It is well-known that some covering properties of a space X can be characterized via corresponding local properties of $C_p(X)$, the set of all real-valued continuous functions $f : X \to \mathbb{R}$ endowed with the topology of pointwise convergence. In particular, we prove that the strong countable fan tightness of $C_p(X)$ does not imply the property α_4 introduced by Arkhangelski.
(Joint work by D.Repovš, B.Tsaban and L.Zdomskyy.)

References

[1] W. Hurewicz, *Über Folgen stetiger Functionen*, Fund. Math. **9** (1927), 193-204;

[2] W. Just, A.W. Miller, M. Scheepers, P.J. Szeptycki, *The combinatorics of open covers II*, Topology Appl. **73** (1996), 241–266.