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Geometric Topology of Busemann G-Spaces

Abstract: The Busemann conjecture is a long standing problem in differential geometry. The fact that a few simple geometric properties on metric spaces can imply the rich structure of a manifold is both remarkable and highly useful. In mathematics, manifolds are generally the spaces of choice. Therefore, it is most useful to know when one is dealing with a manifold.

Herbert Busemann first developed the notion of a geodesic space or G-space in his classic text "The Geometry of Geodesics". His goal was to create an analogue of differential geometry in metric spaces having no predesignated analytic structure. A G-space is defined as a metric space (X, d) that satisfies four simple axioms:

(1) **Menger Convexity** Given distinct points $x, y \in X$, there is a point $z \in X - \{x, y\}$ so that $d(x, z) + d(z, y) = d(x, y)$.
(2) **Finite Compactness** Every d-bounded infinite set has an accumulation point.
(3) **Local Extendibility** To every $w \in X$, there is a positive radius ρ_w, such that for any pair of distinct $x, y \in \text{int } B(w, \rho_w)$, there is $z \in \text{int } B(w, \rho_w) - \{x, y\}$ such that $d(x, y) + d(y, z) = d(x, z)$.
(4) **Uniqueness of the Extension** Given distinct $x, y \in X$, if there are points $z_1, z_2 \in X$ for which both $d(x, y) + d(y, z_i) = d(x, z_i)$ for $i = 1, 2$, and $d(y, z_1) = d(y, z_2)$ hold, then $z_1 = z_2$.

Busemann conjectured that all n-dimensional G-spaces, $n < \infty$, are n-manifolds. The $n = 2$ case was proved in 1955, by Busemann, using purely geometrical techniques. In 1968, a polish mathematician, Krakus, showed the conjecture to be true for $n = 3$ by applying the a topological 2-sphere recognition theorem of Borsuk. In 1993, Paul Thurston proved the conjecture for $n = 4$ by applying elements of metric geometry, algebraic topology, and modern decomposition theory. In 2002 Berestovskii proved the special case of the Busemann Conjecture for Busemann G-spaces that have Alexandrov curvature bounded above. The Busemann conjecture remains unsolved for dimensions $n \geq 5$ in the general case.

The Busemann Conjecture is a special case of the Bing-Borsuk Conjecture. The Bing-Borsuk Conjecture states that finite dimensional homogeneous spaces are manifolds. The Busemann conjecture is also an example of an application of the Moore Conjecture which states

that any resolvable space X has the property that $X \times \mathbb{R}$ is a manifold. Such spaces are called codimension one manifold factors.

If the Busemann conjecture is true, then it strengthens the evidence that the Bing-Borsuk and Moore Conjectures are true and may provide further insights into the proofs of these conjectures. However, a negative result for the Busemann conjecture would also settle the Bing-Borsuk conjecture in the negative. A negative result for the Busemann conjecture plus a positive result for the resolvability of Busemann G-spaces would settle the Moore conjecture in the negative.

This talk will be presented in three parts:

Part 1. History and classical results.
Part 2. Challenges in the high dimensional cases.
Part 3. Implications and relationships to other manifolds recognition problems.