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Geometry and Topology of Busemann's G-spaces

Abstract: The Busemann's G-space (shortly BG-space) can be described briefly as locally compact (metrically) complete inner metric space with properties of local extendability of shortest arcs (metric segments) and uniqueness of their extensions. The letter G relates to the word "geodesic".

The speaker shall give a survey of the following results.

Let X be any BG-space and $B := B(x,r) \subset X$ is a closed ball with a center x of radius $r > 0$. Then one can prove relatively easily the following statements: 1) B is compact; 2) X is separable; 3) any two points in X can be joined by shortest arc; 4) X is geodesically complete; 5) if r is sufficiently small then a) B is homeomorphic to the cone over its boundary sphere $S := S(x,r)$ with the vertex x, b) for any two points x, y in the corresponding open ball $U := U(x,r)$, there is a homeomorphism h of B onto itself, identical on S, such that $h(x) = y$, c) for any point $z \in S$, $S - \{z\}$ is contractible. As a corollary of above properties, 6) X is arc-wise connected, 7) X is locally contractible, 8) X is topologically homogeneous.

If X has a finite topological dimension $n \geq 1$ then it is well-known that 6) and 7) imply that A) X is ENR (Euclidean neighborhood retract); 5), a),b) imply that B) X is the so-called Kosiński r-space; 5), c) implies that C) for any point $z \in S$, $S - \{z\}$ is AR (absolute retract). Then B) and the Lee theorem (1963) imply that D) X is a homological n-manifold. In turn, using 5), a),c), excision axiom, and exact homological sequences (for singular homologies) for pairs $(B, B - \{x\})$ and $(S, S - \{z\})$, one easily gets that S is a homological $(n-1)$-manifold.

Unfortunately, now it is unknown whether any BG-space has a finite topological dimension. As far as the speaker knows, unique reasonable result in this direction is his theorem (1977) that any $BG-$space which γ) contains at least one region such that any closed ball B, containing in the region, is (metrically) convex (i.e. any segment $[y,z] \subset B$ if $y, z \in B$), has a finite topological dimension. In particular, this is true for any $BG-$space X with either of the following property: α) X has non-positive curvature in Busemann sense, β) X has the curvature bounded from above or below in A.D.Aleksandrov sense. Let us remark that in general case BG-space doesn't have property γ) even it is a topological manifold (Busemann, Phadke, Gribanova-Zubareva). The speaker shall explain simply what mean these curvature properties. Really, the speaker proved (2002, 1994) that X admits a structure of smooth manifold in the case β).

If B is a sufficiently small closed and convex ball of positive radius in $BG-$space X and B is n-dimensional manifold with boundary then B is $n-$cell (a corollary of statements by D.Rolfsen, F.Toranzos).

Nobody knows whether any BG-space X with topological dimension $n \geq 5$ has DDP (disjoint discs property). But $X \times \mathbb{R}$ admits a natural structure of Busemann $G-$space and with respect to this structure it has DDP and some other properties.