Unimodal Category

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Outline

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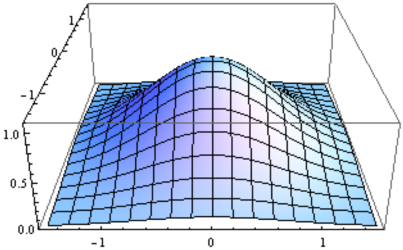
Motivation

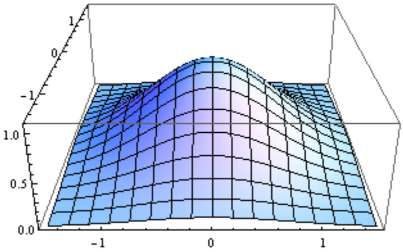
Statistics

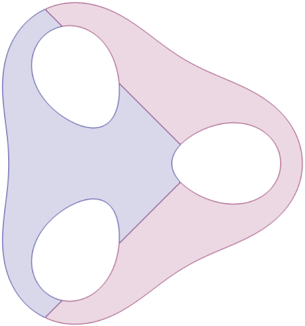


Motivation

Lusternik-Schnirelmann Category

Statistics





Unimodal Category

Definition (Baryshnikov & Ghrist, 2007)

Continuous function $u : X \to [0, \infty)$ is *unimodal* if $u^{-1}[c, \infty)$ are contractible for $0 < c \leq M$ and empty for $c > M$.

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Definition (Baryshnikov & Ghrist, 2007)

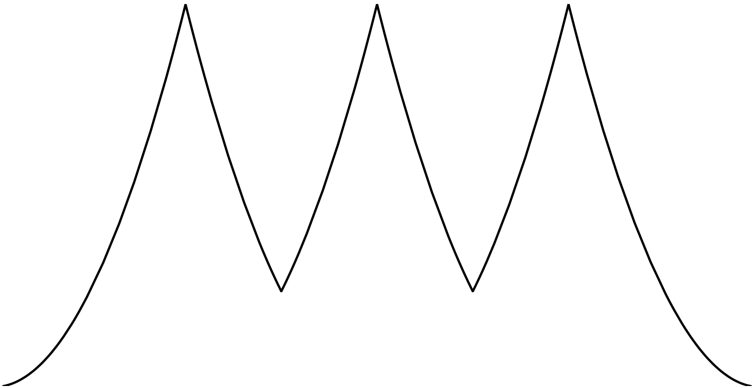
Let $p \in (0, \infty)$. Then

$$\mathbf{ucat}^p(f) = \min\{n \in \mathbb{N}_0 \mid f = (\sum_{i=1}^{n} u_i^p)^{\frac{1}{p}}, u_i \text{ unimodal}\}$$

and

$$\mathbf{ucat}^\infty(f) = \min\{n \in \mathbb{N}_0 \mid f = \max_{1 \leq i \leq n} u_i, u_i \text{ unimodal}\}.$$

Unimodal Category



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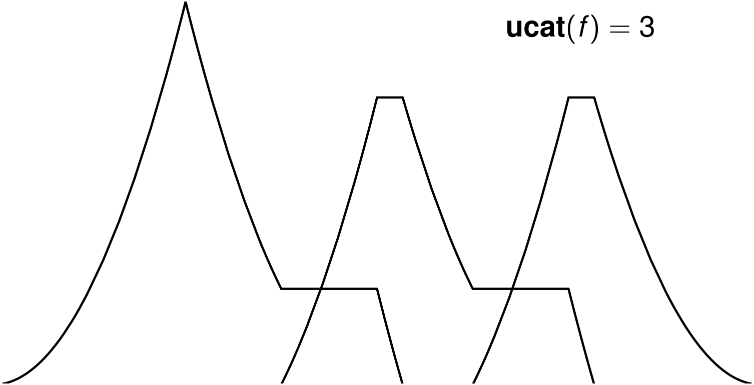

$$\mathbf{ucat}(f) = 3$$

Outline

1. Unimodal Category
   - Basic Definitions
   - Total Variation
   - Real Line & Circle
2. Monotonicity Conjecture
   - Proof for $\mathbb{R}$ and S^1
   - Counterexamples
   - If Morse-Smale Graph $=$ Tree
3. Approximate Nerve Theorem
   - The Nerve Theorem
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Definitions

Total variation:

$$V(f;[a,b]) = \sup \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

Positive variation:

$$V^{+}(f;[a,b]) = \sup \sum_{i=1}^{n} \max\{0, f(x_i) - f(x_{i-1})\},$$

Negative variation:

$$V^{-}(f;[a,b]) = \sup \sum_{i=1}^{n} \max\{0, f(x_{i-1}) - f(x_i)\}.$$

Jordan Decomposition

Theorem

Suppose $f : J \to \mathbb{R}$ is of bounded variation. Then f can be expressed as the difference $f = g - h$ of two increasing functions $g, h : J \to \mathbb{R}$.

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Proof.

Without loss of generality, $\lim_{x \to -\infty} f(x) = 0$. Now simply take $g(x) = V^+(f; J \cap (-\infty, x))$ and $h(x) = V^-(f; J \cap (-\infty, x))$. □

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Forced-Max Condition

Definition (Baryshnikov & Ghrist, 2007; G, 2017)

An interval (x, y) is called **forced-max** (with respect to f) if

$$V^{-}(f; (x, y)) > f(x).$$

Let $M(f)$ be the maximum number of disjoint forced-max intervals (w.r.t. f).

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Theorem (Baryshnikov & Ghrist, 2007; G, 2017)

If $f : \mathbb{R} \to [0, \infty)$ is compactly supported, then

$$\mathbf{ucat}(f) = M(f)$$

holds.

Decomposition Theorem

Theorem (Baryshnikov & Ghrist, 2007; G, 2017)

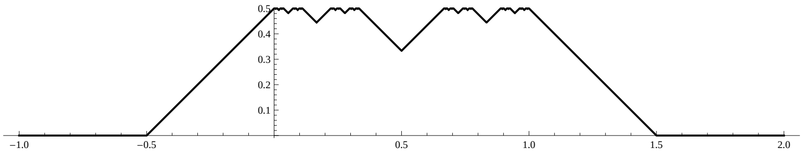
A minimal unimodal decomposition of $f : \mathbb{R} \to [0, \infty)$ *is given by*

$$\begin{aligned} x_0 &= -\infty, \\ x_i &= \inf\{x \mid V^-(f;(x_{i-1}, x)) > f(x_{i-1})\}, \qquad i = 1, \ldots, n, \\ x_{n+1} &= \infty. \end{aligned}$$

and

$$u_i(x) = \begin{cases} 0; & x \le x_{i-1}, \\ g(x) - g(x_{i-1}); & x \in [x_{i-1}, x_i], \\ h(x_{i+1}) - h(x); & x \in [x_i, x_{i+1}], \\ 0; & x \ge x_{i+1}. \end{cases}$$

Decomposition Theorem



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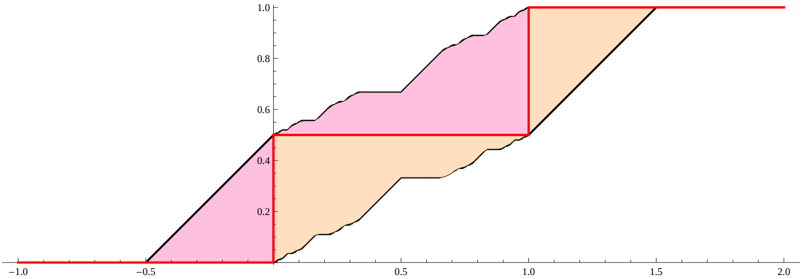

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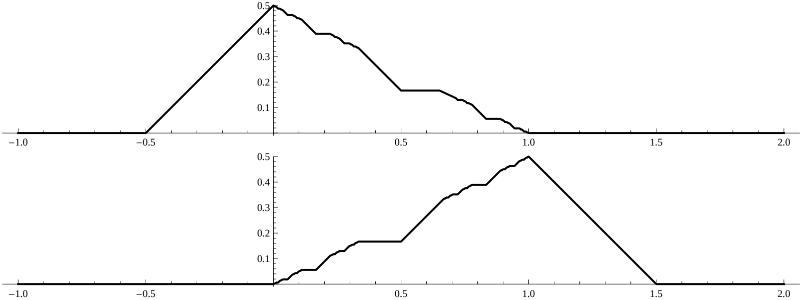

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Monotonicity Conjecture

Conjecture (Baryshnikov & Ghrist, 2007)

Suppose $f : X \to [0, \infty)$ *and* $0 < p_1 < p_2 \leq \infty$. *Then* $\mathbf{ucat}^{p_1}(f) \leq \mathbf{ucat}^{p_2}(f)$.

Proof for $\mathbb{R}$ and S^1

Theorem (G, 2017)

Suppose $X = \mathbb{R}$ *or* $X = S^1$, $f : X \to [0, \infty)$ *and* $0 < p_1 < p_2 \leq \infty$. *Then* $\mathbf{ucat}^{p_1}(f) \leq \mathbf{ucat}^{p_2}(f)$.

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Proof.

Using the Karamata inequality, we can show

$$V^-(f^{p_1}; [a, b]) > f(a)^{p_1} \implies V^-(f^{p_2}; [a, b]) > f(a)^{p_2}.$$

□

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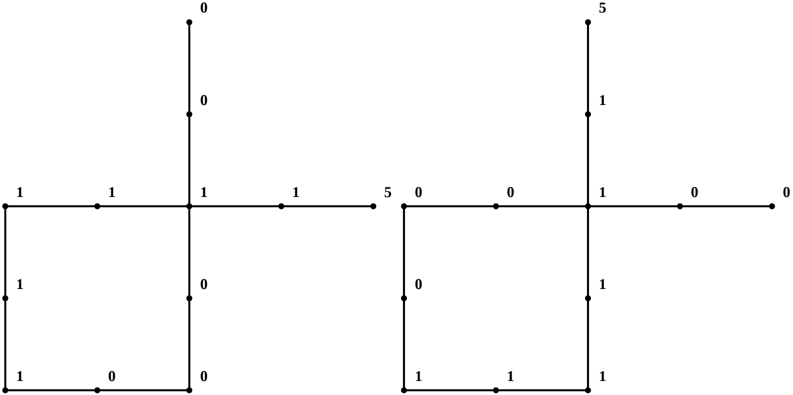
Graphs
(G, 2017)



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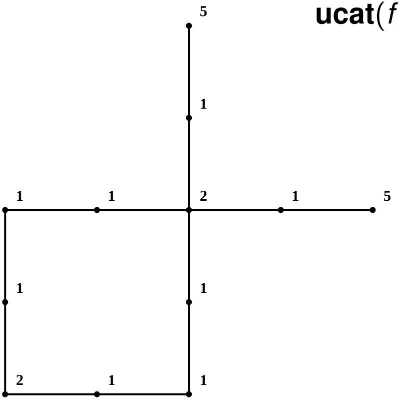

$\mathbf{ucat}(f) = 2$

Graphs
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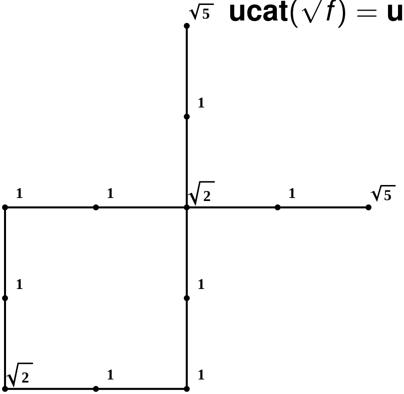
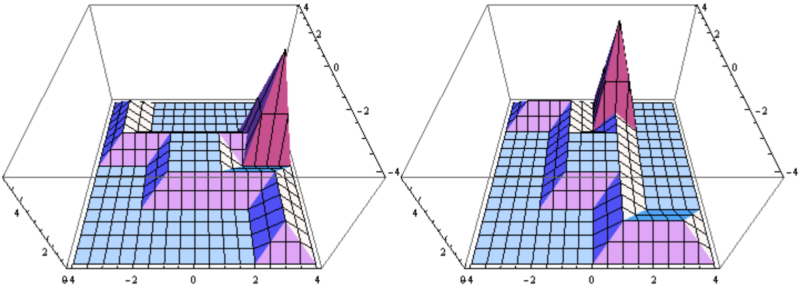
$$\mathbf{ucat}(\sqrt{f}) = \mathbf{ucat}^{\frac{1}{2}}(f) = 3$$



Plane
(G, 2017)



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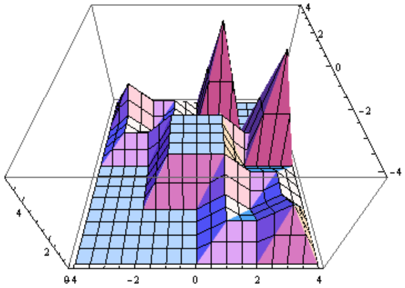

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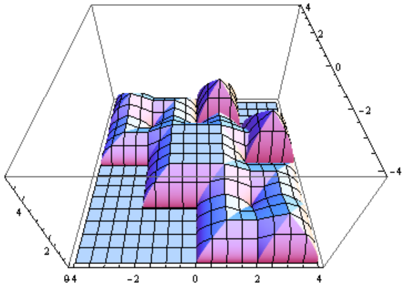

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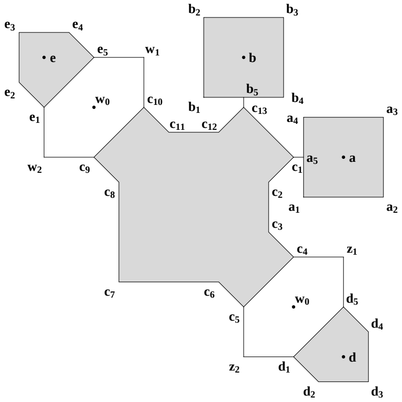
e3
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c13
a4
a3
a5
a
c1
w2
c9
c8
c2
a1
a2
c3
c4
z1
c7
c6
w0
d5
c5
d4
d
z2
d1
d2
d3

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Morse-Smale Graph = Tree $\implies$ Monotonicity

Theorem (G, 2017)

If $f : \mathbb{R}^2 \to [0, \infty)$ is nonresonant and Morse-Smale graph of f is a tree, then $\mathbf{ucat}^p(f)$ is monotone in p.

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- For functions of this kind, **ucat** has a characterization using path values (Hickok, Villatoro & Wang, 2012).

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- Let $g = f^{p_1}$ and $p = \frac{p_2}{p_1}$.
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- Using norm inequalities, $\sum_{i=1}^{n} \mathrm{PV}(x_i, x)^p > g(x)^p$.

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- Using norm inequalities, $\sum_{i=1}^{n} \mathrm{PV}(x_i, x)^p > g(x)^p$.
- Conclude that $\mathbf{ucat}(g^p) \leq n$.

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(Homological) Nerve Theorem

Theorem (Borsuk, 1948)

If $\mathcal{V}$ is an open cover of a paracompact space Y such that every nonempty intersection of finitely many sets in $\mathcal{V}$ is contractible, then Y is homotopy equivalent to the nerve $\mathcal{N}(\mathcal{V})$.

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Theorem (Leray, 1945)

If $\mathcal{U}$ is a cover by subcomplexes of a simplicial complex X such that every nonempty intersection of finitely many sets in $\mathcal{U}$ is acyclic, then

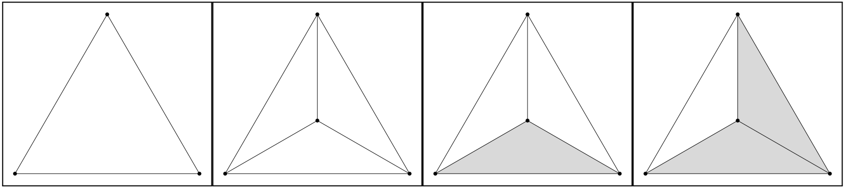
$$\mathrm{H}_*(X) \cong \mathrm{H}_*(\mathcal{N}(\mathcal{U})),$$

where $\mathcal{N}(\mathcal{U})$ is the nerve of $\mathcal{U}$.

Outline

Filtrations

Filtration is given by $f : X \to \mathbb{Z}$.



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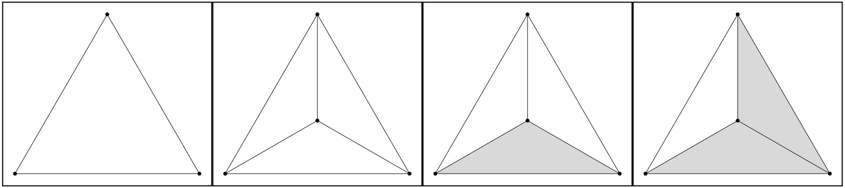

Persistent homology:

$$\mathrm{H}_0(X): \quad \ldots \to \Bbbk \to \Bbbk \to \Bbbk \to \Bbbk \to \ldots$$
$$\mathrm{H}_1(X): \quad \ldots \to \Bbbk \to \Bbbk^3 \to \Bbbk^2 \to \Bbbk \to \ldots$$

Persistence Module

Persistent homology can be understood as a functor $V : (\mathbb{Z}, \leq) \to$ **Vect** or a $\Bbbk[t]$-module.

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Isomorphic categories:

$$\mathbf{Vect}^{(\mathbb{Z}, \leq)} \cong \mathbf{Mod}_{\mathbb{k}[t]}$$

Interleaving

Filtrations $f, g : X \to \mathbb{Z}$ with $\|f - g\|_\infty \leq \varepsilon \implies$ their homologies are ε-interleaved $\Bbbk[t]$-modules.

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Definition

$\mathbb{k}[t]$-modules M and N are ε-interleaved if there is a pair of ε-morphisms $f : M \xrightarrow{\varepsilon} N$ and $g : N \xrightarrow{\varepsilon} M$ such that

$$g(f(m)) = t^{2\varepsilon} m \qquad \text{and} \qquad f(g(n)) = t^{2\varepsilon} n.$$

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This yields a metric between isomorphism classes of modules:

$$d_I(M, N) = \min\{\varepsilon \in \mathbb{N}_0 \mid M \overset{\varepsilon}{\sim} N\}.$$

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Approximate Nerve Theorem

Theorem (G & Škraba, 2016)

Let $D = \dim \mathcal{N}(\mathcal{U})$, $\Delta = \dim X$ *and* $Q = \min(D, \Delta)$.
If $\mathcal{U}$ *is an* ε*-acyclic cover of* X *and* $D < \infty$, *we have*

$$\mathrm{H}_*(X) \overset{2(Q+1)\varepsilon}{\sim} \mathrm{H}_*(\mathcal{N}).$$

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Naively, we obtain $\mathrm{H}_*(X) \overset{(4D+2)\varepsilon}{\sim} \mathrm{H}_*(\mathcal{N})$.
To prove tight bound, we introduce left and right interleavings.

Summary

- For functions on the real line or the circle, the notion of **ucat** is intimately connected to the notion of total variation.
- The monotonicity conjecture does not hold in general. Nontrivial cycles in superlevel sets can be used to construct counterexamples.
- For approximately acyclic covers, there is an approximate nerve theorem. The approximation bounds can be precisely estimated.

Open Questions

- Is there a cohomological approach to **ucat**?
- Does monotonicity hold for multimodal functions?
- In what ways can interleavings be decomposed into left and right interleavings?

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Thank you for your attention!