Axiomatization of geometry employing group actions and topology

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3.6.2015

Abstract

The aim of the talk is to outline a new axiomatization of planar geometry by reinterpreting the original axioms of Euclid. The basic concept is still that of a line segment but its equivalent notion of betweenness is viewed as a topological, not a metric concept. That leads quickly to the notion of connectedness without any need to dwell on the definition of topology. In our approach line segments must be connected. Lines and planes are unified via the concept of separation: lines are separated into two components by each point, planes contain lines that separate them into two components as well. We add a subgroup of bijections preserving line segments and establishing unique isomorphism of basic geometrical sets, and the axiomatic structure is complete. Of fundamental importance is the Fixed Point Theorem that allows for creation of the concepts of length and congruency of line segments.

The resulting structure is much more in sync with modern science than other axiomatic approaches to planar geometry. For instance, it leads naturally to the Erlangen Program in geometry. Our Conditions of Homogeneity and Rigidity have two interpretations. In physics, they correspond to the basic tenet that independent observers should arrive at the same measurement and are related to boosts in special relativity. In geometry, they mean uniqueness of congruence for certain geometrical figures.

Euclid implicitly assumes the concepts of length and angle measure in his axioms. Our approach is to let both of them emerge from axioms. Euclid obfuscates the fact that to compare lengths of line segments one needs rigid motions beforehand. Our system of axioms of planar geometry rectifies that defect of all current axiomatic approaches to planar geometry (of Hilbert and Tarski).

Another thread of the paper is the introduction of boundary at infinity, an important concept of modern mathematics, and linking of Pasch Axiom to endowing boundaries at infinity with a natural relation of betweenness. That way spherical geometry can be viewed as geometry of boundaries at infinity.